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MATROID THEORY AND CHERN-SIMONS

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Abstract

It is shown that matroid theory may provide a natural mathematical framework for a duality symmetries not only for quantum Yang-Mills physics, but also for M-theory. Our discussion is focused in an action consisting purely of the Chern-Simons term, but in principle the main ideas can be applied beyond such an action. In our treatment the theorem due to Thistlethwaite, which gives a relationship between the Tutte polynomial for graphs and Jones polynomial for alternating knots and links, plays a central role. Before addressing this question we briefly mention some important aspects of matroid theory and we point out a connection between the Fano matroid and D=11 supergravity. Our approach also seems to be related to loop solutions of quantum gravity based in Ashtekar formalism.

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1.- INTRODUCTION

In the last few years, duality has been a source of great interest to study nonperturbative, as well as perturbative, dynamics of superstrings [1] and supersymmetric Yang-Mills [2]. In fact, duality is the key physical concept that relates the five known superstring theories in 9+1 dimensions (i.e. nine space and one time), Type I, Type IIA, Type IIB, Heterotic SO(32) and Heterotic $E_8\times$ E_8, which may now be understood as different manifestations of one underlying unique theory called M-theory [3]-[9]. However, dualities are still a mystery and up to now a general understanding how these dualities arises is missing. Nevertheless, just as the equivalence principle is a basic principle in general relativity, the recent importance of dualities in gauge field theories and string theories strongly suggest a duality principle as a basic principle in M-theory. In this sense, what it is needed is a mathematical framework to support such a duality principle.

M-theory is defined as a 10+1 dimensional theory arising as the strong-coupling limit of type IIA string theory. Essentially, M-theory is a non-pertubative theory and in addition to the five superstring theories it describes supermembranes [10], 5-branes [11] and D=11 supergravity [12]. Although the complete M-theory is unknown there are two main proposed routes to construct it. One is the N=(2,1) superstring theory [13] and the other M(atrix)-theory [14]. Martinec [15] has suggested that these two scenarios may, in fact, be closely related. This scenario has been extended [16] to include dualities involving compactifications on timelike circles as well as spacelike circles ones. In particular, it has been shown that T-duality on a timelike circle takes type IIA theory into a type IIB* theory and type IIB* theory into a type IIA theory and that the strong-coupling limit of type IIA* is a theory in 9+2 dimensional theory, denoted by M*.

More recently, Khoury and Verlinde [17] have shed some new light on the old idea of open/closed string duality [18]. This duality is of special interest because emphasizes the idea that closed string dynamics (gravity) is dual to open string dynamics (gauge theory).

Two previous examples on this direction are matrix theory [14], where gravity arises as an effect of open string quantum fluctuations and Maldacena's conjecture [19] that anti-deSitter supergravity is in some sense dual to supersymmetric gauge theory.

Thus, just as the tensor theory makes mathematical sense of the postulate of relativity "the laws of physics are the same for every observer", we are pursuing the possibility that the mathematical formalism necessary to make sense of a duality principle in M-theory is matroid theory [20]. This theory is a generalization of matrices and graphs and , in contrast to graphs in which duality can be defined only for planar graphs, it has the remarkable property that duality can be defined for every matroid. Since M(atrix)-theory and N=(2,1) superstrings have had an important success on describing some essential features of M-theory a natural question is to see whether matroid theory is related to these two approaches. As a first step in this direction we may attempt to see if matroid theory is linked somehow to D=11 supergravity which is a common feature of both formalism. In fact, it has been shown [21] that the Fano matroid and its dual are closely related to Englert's compactification [22] of D=11 supergravity. This result is physically interesting because such a relation allows the connection between the fundamental Fano matroid or its dual [23] and octonions which, at the same time, are one of the alternative division algebras [24]. It is worth mentioning that some time ago Blencowe and Duff [25] raised the question whether the four forces of nature correspond to the four divisions algebras.

In this work, we make further progress on this program. Specifically, we find a route to incorporate matroid theory in quantum Yang-Mills in the context of Chern-Simons action. Our mechanism is based on a theorem due to Thistlethwaite [26] which connect the Jones polynomial for alternating knots with the Tutte polynomial for graphs. Since Witten [27] showed that Jones polynomial can be understood in three dimensional terms through a Chern-Simons formalism it became evident that we have a bridge between graphs and Chern-Simons. In this context duality, which is the main subject in graphic matroids, can be associated to Chern-Simons in a mathematical natural way. This connection may transfer important theorems from matroid theory to fundamental physics. For instance, the theorem

due to Whitney [20] that if $M_1, .., M_p$ and $M_1', .., M_p'$ are the components of the matroids M and M' respectively, and if M_i' is the dual of M_i $(i = 1, ..., p)$ then M' is dual of M and conversely, if M and M' are dual matroids then M_i' is dual of M_i may be applied to dual Chern-Simons partition functions. One of the aims of this work is to explain how this can be done.

The plan of this work is as follows. In section 2, we briefly review matroid theory and in section 3 we closely follow the reference [21] to discuss a connection between matroid theory and D=11 supergravity. In section 4, we study the relation between matroid theory and Witten's partition function for knots. Finally, in section 5, we make some final comments.

2.- A BRIEF REVIEW OF MATROID THEORY

In 1935, while working on abstract properties of linear dependence, Whitney [20] introduced the concept of matroid. In the same year, Birkhoff [28] established the connection between simple matroids (also known as combinatorial geometries [29]) and geometric lattices. In 1936, Mac Lane [30] gave an interpretation of matroids in terms of proyective geometry. And an important progress to the subject was given in 1958 by Tutte [23] who introduced the concept of homotopy for matroids. At present, there is a large body of information about matroid theory. The reader interested in the subject may consult the excellent books on matroid theory by Welsh [31], Lawler [32] and Tutte [33]. We also recommend the books of Wilson [34], Kung [35] and Ribnikov [36].

An interesting feature of matroid theory is that there are many different but equivalent ways of defining a matroid. In this respect, it seems appropriate to briefly review the Whitney's [20] discovery of the matroid concept. While working with linear graphs Whitney noticed that for certain matrices duality had a simple geometrical interpretation quite different than in the case of graphs. Further, he observed that any subset of columns of a matrix is either linearly independent or linearly dependent and that the following two theorems must hold:

(a) any subset of an independent set is independent.

(b) if N_p and N_{p+1} are independent sets of p and p+1 columns respectively, then N_p together with some column of N_{p+1} forms an independent set of p+1 columns.

Moreover, he discovered that if these two statements are taking as axioms then there are examples that do not represent any matrix and graph. Thus, he concluded that a system satisfying (a) and (b) should be a new one and therefore deserved a new name: He called to this kind of system a matroid.

The definition of a matroid in terms of independent sets has been refined and is now expressed as follows: A matroid M is a pair (E,$\mathcal{I}$), where E is a non-empty finite set, and $\mathcal{I}$ is a non-empty collection of subsets of E (called independent sets) satisfying the following properties:

($\mathcal{I}$ *i)* any subset of an independent set is independent;

($\mathcal{I}$ *ii)* if I and J are independent sets with I$\subseteq$ J, then there is an element e contained in J but not in I, such that I$\cup\{e\}$ is independent.

A base is defined to be any maximal independent set. By repeatedly using the property ($\mathcal{I}$ *ii)* it is straightforward to show that any two bases have the same number of elements. A subset of E is said to be dependent if it is not independent. A minimal dependent set is called a circuit. Contrary to the bases not all circuits of a matroid have the same number of elements.

An alternative definition of a matroid in terms of bases is as follows:

A matroid M is a pair (E, $\mathcal{B}$), where E is a non-empty finite set and $\mathcal{B}$ is a non-empty collection of subsets of E (called bases) satisfying the following properties:

($\mathcal{B}$ *i)* no base properly contains another base;

($\mathcal{B}$ *ii)* if B_1 and B_2 are bases and if b is any element of B_1, then there is and element g of B_2 with the property that (B_1-$\{b\}$)$\cup\{g\}$ is also a base.

A matroid can also be defined in terms of circuits:

A matroid M is a pair (E, $\mathcal{C}$), where E is a non-empty finite set, and $\mathcal{C}$ is a collection of a non-empty subsets of E (called circuits) satisfying the following properties.

($\mathcal{C}$ *i*) no circuit properly contains another circuit;

($\mathcal{C}$ *ii)* if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two distinct circuits each containing and element c, then there exists a circuit in $\mathcal{C}_1 \cup \mathcal{C}_2$ which does not contain c.

If we start with any of the three definitions the other two follows as a theorems. For example, it is possible to prove that ($\mathcal{I}$ *)* implies ($\mathcal{B}$) and ($\mathcal{C}$). In other words, these three definitions are equivalent. There are other definitions also equivalent to these three, but for the purpose of this work it is not necessary to consider them.

Notice that even from the initial structure of a matroid theory we find relations such as independent-dependent and base-circuit which suggests duality. The dual of M, denoted by M^*, is defined as a pair (E, $\mathcal{B}^*$), where $\mathcal{B}^*$ is a non-empty collection of subsets of E formed with the complements of the bases of M. An immediate consequence of this definition is that every matroid has a dual and this dual is unique. It also follows that the double-dual M^{**} is equal to M. Moreover, if A is a subset of E, then the size of the largest independent set contained in A is called the rank of A and is denoted by ρ(A). If $M = M_1 + M_2$ and $\rho(M)$ $= \rho(M_1) + \rho(M_2)$ we shall say that M is separable. Any maximal non-separable part of M is a component of M. An important theorem due to Whitney [20] is that if $M_1, .., M_p$ and $M_1^{'}, .., M_p^{'}$ are the components of the matroids M and M' respectively, and if $M_i^{'}$ is the dual of M_i (i = 1,...,p). Then M' is dual of M. Conversely, let M and M' be dual matroids, and let $M_1, .., M_p$ be components of M. Let $M_1^{'}, .., M_p^{'}$ be the corresponding submatroids of M'. Then $M_1^{'}, .., M_p^{'}$ are the components of M', and $M_i^{'}$ is dual of M_i.

3.- MATROID THEORY AND SUPERGRAVITY

Among the most important matroids we find the binary and regular matroids. A matroid is binary if it is representable over the integers modulo two. Let us clarify this definition. An important problem in matroid theory is to see which matroids can be mapped in some set of vectors in a vector space over a given field. When such a map exists we are speaking of a coordinatization (or representation) of the matroid over the field. Let GF(q) denote a

finite field of order q. Thus, we can express the definition of a binary matroid as follows: A matroid which has a coordinatization over GF(2) is called binary. Furthermore, a matroid which has a coordinatization over every field is called regular. It turns out that regular matroids play a fundamental role in matroid theory, among other things, because they play a similar role that planar graphs in graph theory [34]. It is known that a graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$. The analogue of this theorem for matroids was provided by Tutte [23]. In fact, Tutte showed that a matroid is regular if and only if is binary and includes no Fano matroid or the dual of this. In order to understand this theorem it is necessary to define the Fano matroid. We shall show that the Fano matroid may be connected with octonions which, in turn, are related to the Englert's compactification of D=11 supergravity.

A Fano matroid F is the matroid defined on the set E={1,2,3,4,5.6.7} whose bases are all those subsets of E with three elements except f_1 ={1,2,4}, f_2 ={2,3,5}, f_3 = {3,4,6}, f_4 ={4,5,7}, f_5 ={5,6,1}, f_6 ={6,7,2} and f_7 ={7,1,3}. The circuits of the Fano matroid are precisely these subsets and its complements. It follows that these circuits define the dual F^* of the Fano matroid.

Let us write the set E in the form $\mathcal{E}=\{e_1, e_2, e_{3,}e_{4,}e_5, e_6, e_7\}$. Thus, the subsets used to define the Fano matroid now become $f_1 = \{e_1, e_2, e_4\}$, $f_2\{e_2, e_3, e_5\}$, $f_3\{e_3, e_4, e_6\}$, $f_4\{e_4, e_5, e_7\}$, $f_5\{e_5, e_6, e_1\}$, $f_6\{e_6, e_7, e_2\}$ and f_7 $\{e_7, e_1, e_3\}$. The central idea is to identify the quantities e_i, where $i = 1, ...,7$, with the octonionic imaginary units. Specifically, we write an octonion q in the form $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 + q_4e_4 + q_5e_5 + q_6e_6 + q_7e_7$, where q_0 and q_i are real numbers. Here, e_0 denotes the identity. The product of two octonions can be obtained with the rule:

$$e_ie_j = -\delta_{ij} + \psi_{ij}^k e_k, \tag{1}$$

where δ_{ij} is the Kronecker delta and $\psi_{ijk} = \psi_{ij}^l \delta_{lk}$ is the fully antisymmetric structure constants, with $i, j, k = 1, ..., 7$. By taking the ψ_{ijk} equals 1 for one of the seven combinations f_i we may derive all the values of ψ_{ijk}.

The octonion (Cayley) algebra is not associative, but alternative. This means that the basic associator of any three imaginary units is

$$(e_i, e_j, e_k) = (e_i e_j)e_k - e_i(e_j e_k) = \varphi_{ijkm} e_m, \tag{2}$$

where φ_{ijkl} is a fully antisymmetric four index tensor. It turns out that φ_{ijkl} and ψ_{ijk} are related by the expression

$$\varphi_{ijkl} = (1/3!)\epsilon_{ijklmnr}\psi_{mnr}, \tag{3}$$

where $\epsilon_{ijklmnr}$ is the completely antisymmetric Levi-Civita tensor, with $\epsilon_{12...7} = 1$. It is interesting to note that given the numerical values f_i for the indices of ψ_{mnr} and using (3) we get the other seven subsets of E with four elements of the dual Fano matroid F^*. For instance, if we take f_1 then we have ψ_{124} and (3) gives φ_{3567} which leads to the circuit subset $\{3, 5, 6, 7\}$.

We would like now to relate the above structure to the Englert's octonionic solution [22] of eleven dimensional supergravity. First, let us introduce the metric

$$g_{ab} = \delta_{ij} h^i_a h^j_b, \tag{4}$$

where $h^i_a = h^i_a(x^c)$ is a sieben-bein, with $a, b, c = 1, ..., 7$. Here, x^c are a coordinates patch of the geometrical seven sphere S^7. The quantities ψ_{ijk} can now be related to the S^7 torsion in the form

$$T_{abc} = R_0^{-1}\psi_{ijk} h^i_a h^j_b h^k_c, \tag{5}$$

where R_0 is the S^7 radius. While the quantities φ_{ijkl} can be identified with the four index gauge field F_{abcd} through the formula

$$F_{abcd} = R_0^{-1}\varphi_{ijkl} h^i_a h^j_b h^k_c h^l_d. \tag{6}$$

Furthermore, it is possible to prove that the Englert's 7-dimensional covariant equations are solve with the identification $F_{abcd} = \lambda T_{[abc;d]}$, where λ is a constant. Therefore, $\lambda T_{abc} = A_{abc}$ is the fully antisymmetric gauge field which is a fundamental object in 2-brane theory [6].

It is important to mention that in the Englert's solution of D= 11 supergravity the torsion satisfies the Cartan-Schouten equations

$$T_{acd}T_{bcd} = 6R_0^{-2}g_{ab}, \tag{7}$$

$$T_{ead}T_{dbf}T_{fce} = 3R_0^{-2}T_{abc.} \tag{8}$$

But as Gursey and Tze [37] noted, these equations are mere septad-dressed, i.e. covariant forms of the algebraic identities

$$\psi_{ikl}\psi_{jkl} = 6\delta_{ij}, \tag{9}$$

$$\psi_{lim}\psi_{mjn}\psi_{nkl} = 3\psi_{ijk}, \tag{10}$$

respectively. It is worth mentioning that Englert solution realizes the riemannian curvatureless but torsion-full Cartan-geometries of absolute parallelism on S^7.

So, we have shown that the Fano matroid is closely related to octonions which at the same time are an essential part of the Englert's solution of absolute parallelism on S^7 of D=11 supergravity. The Fano matroid and its dual are the only minimal binary irregular matroids. We know from Hurwitz theorem (see reference [24]) that octonions are one of the alternative division algebras (the others are the real numbers, the complex numbers and the quaternions). While among the only parallelizable spheres we find S^7 (the other are the spheres S^1 and S^3 [38]). This distinctive and fundamental role played by the Fano matroid, octonions and S^7 in such different areas in mathematics as combinatorial geometry, algebra and topology respectively lead us to believe that the relationship between these three concepts must have a deep significance not only in mathematics, but also in physics. Of course, it is known that the parallelizability of S^1, S^3 and S^7 has to do with the existence of the complex numbers, the quaternions and the octonions respectively (see reference [39]). It is also known that using an algebraic topology called K-theory [40] we find that the only dimensions for division algebras structures on Euclidean spaces are 1, 2, 4, and 8. We can

add to these remarkable results another fundamental concept in matroid theory; the Fano matroid.

4.- MATROID THEORY AND CHERN-SIMONS

Before going into details, it turns out to be convenient to slightly modify the notation of the previous section. In this section, we shall assume that the Greek indices $\alpha, \beta, ..., etc$ run from 0 to 3, the indices $i, j, ..., etc$ run from 0 to 2 and finally the indices $a, b, ..., etc$ take values in the rank of a compact Lie Group G. Further, we shall denote a compact oriented four manifold as M^4.

Consider the second Chern class action

$$S = \frac{k}{16\pi} \int_{M^4} \epsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^b_{\alpha\beta}\ g_{ab}, \tag{11}$$

with the curvature given by

$$F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + C^a_{bc} A^b_\alpha A^c_\beta. \tag{12}$$

Here g_{ab} is the Killling-Cartan metric and C^a_{bc} are the completely antisymmetric structure constants associated to the compact simple Lie group G. The action (11) is a total derivative and leads to the Chern-Simons action

$$S_{CS} = \frac{k}{4\pi} \int_{M^3} \{\epsilon^{ijk} (A^a_i (\partial_j A^b_k - \partial_k A^b_j) g_{ab} + \frac{2}{3} C_{abc} A^a_i A^b_j A^c_k)\}, \tag{13}$$

where $M^3 = \partial M^4$ is a compact oriented three dimensional manifold. In a differential forms notation (13) can be rewritten as follows:

$$S_{CS} = \frac{k}{2\pi} \int_{M^3} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{14}$$

where $A = A^a_i T_a dx^i$, with T_a the generators of the Lie algebra of G.

Given a link L with r components and irreducible representation ρ_r of G, one for each component of the link, Witten [27] defines the partition function

$$Z(L,k) = \int D\mathcal{A} \exp(iS_{cs}) \prod_{r=1}^{n} W(L_r,\rho_r), \tag{15}$$

where $W(C_i, \rho_i)$ is the Wilson line

$$W(L_r, \rho_r) = Tr_{\rho_r} P \exp(\int_{L_r} A_i^a T_a dx^i). \tag{16}$$

Here the symbol P means the path-ordering along the knots L_r.

If we choose $M^3 = S^3$, $G = SU(2)$ and ρ_r=C^2 for all link components then the Witten's partition function (15) reproduces the Jones polynomial

$$Z(L,k) = V_L(t), \tag{17}$$

where

$$t = e^{\frac{2\pi i}{k}}. \tag{18}$$

Here $V_L(t)$ denotes the Jones polynomial satisfying the skein relation:

$$t^{-1} V_{L_+} - t V_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}}) V_{L_0}, \tag{19}$$

where L_+, L_- and L_0 are the standard notation for overcrossing, undercrossing and zero crossing.

Now, lets pause about the relation between the knots and Chern-Simons term and let us discuss the Tutte polynomial. To each graph $\mathcal{G}$, we associate a polynomial $T_{\mathcal{G}}(x, x^{-1})$ with the property that if $\mathcal{G}$ is composed solely of isthmus and loops then $T_{\mathcal{G}}(x, x^{-1}) = x^I x^{-l}$, where I is the number of isthmuses and l is the number of loops. The polynomial $T_{\mathcal{G}}$ satisfies the skein relation

$$T_{\mathcal{G}} = T_{\mathcal{G}'} + T_{\mathcal{G}''}, \tag{20}$$

where $\mathcal{G}'$ and $\mathcal{G}''$ are obtained by delating and contracting respectively an edge that is neither a loop nor an isthmus of $\mathcal{G}$.

There is a theorem due to Thistlethwaite [26] which assures that if L is an alternating link and $\mathcal{G}(L)$ the corresponding planar graph then the Jones polynomial $V_L(t)$ is equal to the Tutte polynomial $T_{\mathcal{G}}(-t, -t^{-1})$ up to a sign and factor power of t. Specifically, we have

$$V_L(t) = (-t^{\frac{3}{4}})^{w(L)} t^{\frac{-(r-n)}{4}} T_{\mathcal{G}}(-t, -t^{-1}) \tag{21}$$

where $w(L)$ is the writhe and r and n are the rank and the nullity of $\mathcal{G}$ respectively. Here $V_L(t)$ is the Jones polynomial of alternating link L.

On the other hand, a theorem due to Tutte allows to compute $T_{\mathcal{G}}(-t, -t^{-1})$ from the maximal trees of $\mathcal{G}$. In fact, Tutte proved that if $\mathcal{B}$ denotes the maximal trees in a graph $\mathcal{G}$, $i(B)$ denotes the number of internally active edges in $\mathcal{G}$ and $e(B)$ the number the externally active edges in $\mathcal{G}$ (with respect to a given maximal tree $B \epsilon \mathcal{B}$) then the Tutte polynomial is given by the formula

$$T_{\mathcal{G}}(-t, -t^{-1}) = \sum x^{i(B)} x^{-e(B)} \tag{22}$$

where the sum is over all elements of $\mathcal{B}$.

First, note that $\mathcal{B}$ is the collection of bases of $\mathcal{G}$. If we now remember our definition of matroid M in terms of bases discussed in section 2 we note the Tutte polynomial $T_{\mathcal{G}}(-t, -t^{-1})$ computed according to (22) uses the concept of a graphic matroid $M(\mathcal{G})$ defined as the pair (E, $\mathcal{B}$), where E is the set of edges of $\mathcal{G}$. In fact, the elements of $\mathcal{B}$ satisfy the two properties

($\mathcal{B}$ *i)* no base properly contains another base;

($\mathcal{B}$ *ii)* if B_1 and B_2 are bases and if b is any element of B_1, then there is and element g of B_2 with the property that (B_1-$\{b\}$)$\cup$$\{g\}$ is also a base.

which identifies a $M(\mathcal{G})$ as a matroid. With this remarkable connection between the Tutte polynomial and a matroid we have found in fact a connection between the partition function $Z(L,k)$ given in (15) and matroid theory. This is because according to (21) the Tutte polynomial $T_{\mathcal{G}}(-t, -t^{-1})$ are related to the Jones polynomial $V_L(t)$ which at the same time according to (17) are related to the partition function $Z(L,k)$. Specifically, for $M^3 = S^3$, $G = SU(2)$, ρ_r=C^2 for all alternating link components of L, we have the relation

$$Z(L,k) = V_L(t) = (-t^{\frac{3}{4}})^{w(L)} t^{\frac{-(r-n)}{4}} T_{\mathcal{G}}(-t, -t^{-1}). \tag{23}$$

Thus, the matroid (E, $\mathcal{B}$) used to compute $T_{\mathcal{G}}(-t, -t^{-1})$ can be associated not only to $V_L(t)$, but also to $Z(L,k)$.

Now that we have at hand this slightly but important connection between matroid theory and Chern-Simons theory we arre able to transfer information from matroid theory to Chern-Simons and conversely from Chern-Simons to matroid theory. Let us discuss two examples for the former possibility.

First of all, it is known that in matroid theory the concept of duality is of fundamental importance. For example, there is a remarkable theorem that assures that every matroid has a dual. So, the question arises about what are the implications of this theorem in Chern-Simons formalism. In order to address this question let us first make a change of notation $T_{\mathcal{G}}(-t,-t^{-1}) \rightarrow T_{M(\mathcal{G})}(t)$ and $Z(L,k) \rightarrow Z_{M(\mathcal{G})}(k)$. The idea of this notation is to emphasize the connection between matroid theory, Tutte polynomial and Chern- Simons partition function. Consider the planar dual graph $\mathcal{G}^*$ of $\mathcal{G}$. In matroid theory we have $M(\mathcal{G}^*)$ $=M^*(\mathcal{G})$. Therefore, the duality property of the Tutte polynomial

$$T_{\mathcal{G}}(-t,-t^{-1}) = T_{\mathcal{G}^*}(-t^{-1},-t) \tag{24}$$

can be expressed as

$$T_{M(\mathcal{G})}(t) = T_{M^*(\mathcal{G})}(t^{-1}) \tag{25}$$

and consequently from (23) we discover that for the partition function $Z_{M(\mathcal{G})}(k)$ we should have the duality property

$$Z_{M(\mathcal{G})}(k) = Z_{M^*(\mathcal{G})}(-k). \tag{26}$$

This duality symmetry for the partition function $Z_{M(\mathcal{G})}(k)$ is not really new, but is already known in the literature as mirror image symmetry (see, for instance [41], and references quoted there). However, what seems to be new is the way we had derived it.

As a second example let us first mention another theorem due to Withney [20]: If $M_1,..,M_p$ and $M_1^{'},..,M_p^{'}$ are the components of the matroids M and M' respectively, and if $M_i^{'}$ is the dual of M_i $(i=1,...,p)$. Then M' is dual of M. Conversely, let M and M' be dual matroids, and let $M_1,..,M_p$ be components of M. Let $M_1^{'},..,M_p^{'}$ be the corresponding

submatroids of M'. Then $M_1', .., M_p'$ are the components of M', and M_i' is dual of M_i. Thus, according to (26) we find that

$$Z_{M_i(\mathcal{G}_i)}(k) = Z_{M_i'(\mathcal{G}_i)}(-k) \tag{27}$$

if and only if

$$Z_{M(\mathcal{G})}(k) = Z_{M'(\mathcal{G})}(-k), \tag{28}$$

where $\mathcal{G}_i$ are the components of $\mathcal{G}$.

5.- COMMENTS

Motivated by a possible duality principle in M-theory we have started to bring information from matroid theory to fundamental physics. We now have two good examples which indicate that this task makes sense. In the first example, we have found enough evidence for a connection between the Fano matroid and supergravity in D=11. While in the second example, we have found a relation between the graphic matroid and the Witten's partition function for Chern-Simons. This relation is of special importance because leads us to a duality symmetry in the partition function $Z_{M(\mathcal{G})}(k)$. In fact, if there is a duality principle in M-theory we should expect to a have a duality symmetry in the corresponding partition function associated to M-theory.

In this work, we have concentrated in the original connection between Chern-Simons action and knots theory. But it is well known that Chern-Simons formalism and knots connection has a number of extensions [41]. It will be interesting to study such extensions from the point of view of matroid theory. It is also known Chern-Simons formalism is closely related to conformal field theory and this in turn is closely related to string theory. So, it seems that the present work may eventually leads to a connection between matroid theory and string theory. In order to achieve this goal we need to study the relation between matroids and Chern-Simons using signed graphs [42]. This is because general knots and

links (not only alternating) are one to one correspondence with signed planar graphs. This in turn are straightforward connected with Kauffmann polynomials [43] which at the same time leads to the Jones polynomials. But, signed graphs leads to signed matroids. So, one of our future goals will be to find a connection between signed matroids and string theory. Moreover, matrix Chern-Simons theory [44] has a straightforward relation with Matrix-model and non-commutative geometry [45]. So, a natural extension of the present work will be to study the relation between matroid theory and matrix Chern-Simons theory.

An important duality in M-theory is the strong/weak coupling S-duality [46] which provides with one of the most important techniques to study non-perturbative aspects of gauge field theory and string theory. For further work it may also be important to find the relation between the duality symmetry for $Z_{M(\mathcal{G})}(k)$ given in (27) and S-duality.

Besides of the possible connection between M(atroid)-theory and M-theory there is another interesting physical application of the present work. This has to do with loop solutions of quantum gravity based in Ashtekar formalism. It is known that the Witten's partition function provides a solution of the Ashtekar constriants [47]. So, the duality symmetries (27) also applies to such solutions. In other words, it seems that we have also found a connection between matroid theory and loop solutions of quantum canonical gravity.

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